On an Algebraic System of Equations

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We study the following algebraic system of equations

$$P_\alpha(x_1,\ldots,x_n) := P_\alpha(x) := x^\alpha - \sum_{\beta\in I} a_{\alpha,\beta} x^\beta = 0, \quad \alpha \in J, \tag{1}$$

where $I = \{\alpha \in Z_+^n : |\alpha| \le m\}$, $\quad J := J_{m+1} = \{\alpha \in Z_+^n : |\alpha| = m+1\}$.
We consider some $\#I \times \#I$ - matrices: $A_1,\ldots,A_n$. The structure of these matrices is as follows: k rows ($k = \#J_m$) are the coefficients of k polynomials from $\{P_\alpha : \alpha \in J\}$ and each of the remaining rows consists of 0's and one 1.
We prove that the number of distinct solutions to (1) equals to maximal possible $\#I$ if and only if the above matrices are commuting and each of them is semisimple (i.e. they have complete system of eigenvectors).

This gives a characterization of correct systems of points for multivariate Lagrange interpolation. This also reduces the solving of (1) to solving of n univariate algebraic equations: to finding the eigenvalues of the above matrices. In the case when the eigenvalues of some matrix are distinct, the solving of (1) is reducing to solving of only one univariate equation: to finding of these distinct eigenvalues.

The above results remain valid also for more general systems (1): with I being any lower set.